

# Symmetry-protected topological phases of alkaline-earth cold fermionic atoms in one dimension

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(Dated: October 9, 2012)

We investigate the existence of symmetry-protected topological phases in one-dimensional alkaline-earth cold fermionic atoms with general half-integer nuclear spin  $I$  at half filling. Using complementary techniques, we show that  $SU(2)$  topological phases are stabilized where the  $SU(2)$  symmetry stems from the existence of a metastable excited state in alkaline-earth atoms. On top of these phases, we find the emergence of topological phases with enlarged  $SU(2I + 1)$  symmetry which depend only on the nuclear spins degrees of freedom. The main physical properties of the latter phases are further studied using a matrix-product state approach. We find that these phases are symmetry-protected topological phases, with respect to inversion symmetry, when  $I = 1/2, 5/2, 9/2, \dots$ , which is directly relevant to ytterbium and strontium cold fermions.

PACS numbers: 71.10.Pm, 75.10.Pq

Quantum phases of matter with exotic orderings have attracted much interest over the years. Prominent examples are topological phases which do not break any symmetry and cannot be characterized by local order parameters. Though all one-dimensional (1D) gapful phases have short-range entanglement, a topological phase can still be defined in 1D by the presence of a symmetry which protects the phase.<sup>1</sup> Without the symmetry, the phase can be smoothly connected to a trivial gapful phase without any quantum phase transition.<sup>2</sup> In this respect, a topological phase with a symmetry is then called a symmetry-protected topological (SPT) phase.<sup>1,7</sup> The Haldane phase<sup>3</sup> of the spin-1 Heisenberg chain displays striking properties which makes it a paradigmatic example of a SPT phase in 1D. There is a finite gap above a unique ground state with periodic boundary conditions while spin-1/2 edge states are liberated when the chain is open.<sup>4</sup> This phase exhibits also non-local string long-range ordering.<sup>5</sup> These properties of the Haldane phase have been shown to be protected by the presence of three global symmetries: the dihedral group of  $\pi$  rotations along the  $x, y, z$  axes, time-reversal and inversion symmetries.<sup>6</sup>

A recent breakthrough leads to the complete classification of SPT of 1D spin systems.<sup>8</sup> It relies on the determination of the projective representations of the underlying symmetry group which can be obtained using group cohomology. In particular, for spin systems with a high  $SU(N)$  symmetry,  $N$  distinct topological phases are expected from this classification.<sup>9</sup> A natural question is the physical realization of these  $SU(N)$  topological phases starting from realistic fermionic systems.

In this respect, alkaline-earth like fermionic ultracold atoms seem to be very promising since they are the best candidates for the experimental realization of exotic high-symmetry many-body physics.<sup>10,11</sup> In this context, the main interest in those atoms stems from the presence of a ground state  $^1S_0$  (“ $g$ ”) and of a metastable excited state  $^3P_0$  (“ $e$ ”) between which transitions are forbidden. Moreover, both states have zero electronic angular momentum, so that the nuclear spin  $I$  is decoupled from the electronic spin. The nuclear spin-

dependent variation of the scattering lengths is expected to be smaller than  $\sim 10^{-9}$  for the ground state and  $\sim 10^{-3}$  for the  $e$  state, and this results in fermionic systems with an extended  $SU(N = 2I + 1)$  symmetry.<sup>10</sup> Such gases of strontium ( $^{87}\text{Sr}$ ) atoms ( $I = 9/2$ ) or ytterbium ( $^{171}\text{Yb}$ ,  $^{173}\text{Yb}$ ) atoms ( $I = 1/2, 5/2$  respectively) have been cooled down to reach the quantum degeneracy.<sup>12,13</sup>

A simple model describing the low-energy properties of alkaline-earth ultracold fermionic atoms loaded into a 1D optical lattice reads as follows:<sup>14</sup>

$$\mathcal{H} = -t \sum_{i,l\alpha} \left( c_{l\alpha,i}^\dagger c_{l\alpha,i+1} + H.c. \right) - \mu \sum_i n_i + \frac{U}{2} \sum_i n_i^2 + J \sum_i [(T_i^x)^2 + (T_i^y)^2] + J_z \sum_i (T_i^z)^2, \quad (1)$$

where  $c_{l\alpha,i}^\dagger$  denotes the fermion creation operator at the  $i^{\text{th}}$  site with orbital index  $l = g, e$ , and nuclear-spin states  $\alpha = 1, \dots, N$ . In Eq. (1),  $n_i = \sum_{l\alpha} c_{l\alpha,i}^\dagger c_{l\alpha,i}$  denotes the occupation number on the site  $i$ , and  $2\vec{T}_i = \sum_{lm\alpha} c_{l\alpha,i}^\dagger \vec{\sigma}_{lm} c_{m\alpha,i}$  is the orbital pseudospin operator ( $\vec{\sigma}$  being the Pauli matrices). On top of the standard  $U(1)_c$  charge symmetry ( $c_{l\alpha,i} \rightarrow e^{i\theta} c_{l\alpha,i}$ ), model (1) features an  $SU(N)$  symmetry ( $c_{l\alpha,i} \rightarrow \sum_{\beta} U_{\alpha\beta} c_{l\beta,i}$ ,  $U$  being an  $SU(N)$  matrix), and a  $U(1)_o$  symmetry in the orbital space spanned by the  $g, e$  states ( $c_{g,e\alpha,i} \rightarrow e^{\pm i\theta} c_{g,e\alpha,i}$ ). The continuous symmetry of model (1) is thus  $U(1)_c \times U(1)_o \times SU(N)$ . If one keeps only the  $g$  state, model (1) corresponds to the  $U(N) = U(1)_c \times SU(N)$  Hubbard model. Its gapped Mott-insulating phases are known to be spatially nonuniform for all commensurate fillings.<sup>15</sup> In this respect, no topological phases can be formed in this one-band model.

In this Letter, we show, by means of complementary methods, that the situation is radically different if one includes the metastable excited state (i.e., the  $e$  state) of alkaline-earth atoms. The interplay between orbital and nuclear-spin degrees of freedom in this system gives rise to the emergence of different 1D insulating topological phases at half filling ( $N$  atoms

per site). Two different classes of topological phases are stabilized when  $N$  is even, i.e.,  $I$  is a half-odd integer. We find Haldane phases with integer orbital pseudospin- $N/2$  and topological phases with enlarged  $SU(N)$  symmetry that we fully characterize. In particular, we investigate the topological protection of these phases with respect to the inversion symmetry by means of a matrix-product state (MPS) approach. It is shown that the topological phases of alkaline-earth cold fermions are SPT phases when  $N/2$  is odd, i.e.,  $I = 1/2, 5/2, 9/2, \dots$ , which is directly relevant to ytterbium and strontium atoms.

**Strong-coupling approach.** A strong-coupling analysis along special lines of model (1) is useful to shed light on the possible existence of exotic Mott-insulating phases. In this respect, we consider the special case  $J = J_z$  where the  $U(1)_o$  orbital symmetry is enlarged to  $SU(2)_o$ . The energy-spectrum for the single-site problem at half-filling ( $\mu = UN$ ) is then labelled by two integers  $p, q$ :

$$E(p, q) = \frac{U}{2}(p+q)(p+q-2N) + \frac{J}{4}(p-q)(p-q+2)$$

$$D(p, q) = \frac{N!(N+1)!(p-q+1)^2}{(N-p)!(N+1-q)!(p+1)!q!}, \quad (2)$$

where  $D$  is the degeneracy,  $n = p + q$  is the number of fermions on one site, and  $T = (p - q)/2$  is the spin of the orbital pseudospin operator  $\vec{T}_i$ . When  $J = 0$ , the model is  $U(2N)$  symmetric. For  $U > 0$ , the lowest energy states have  $n = N$  fermions and states with different  $T$  are degenerate. These states transform in the self-conjugate antisymmetric representations of  $SU(2N)$ . The resulting phase in the strong-coupling regime is known to be a spin-Peierls (SP) phase with bond ordering which breaks spontaneously the translation symmetry.<sup>16,17</sup> In contrast, for  $U < 0$ , the lowest energy states have  $n = 0$  or  $2N$  fermions and a charge-density wave (CDW) phase is stabilized.<sup>18</sup> A second interesting line is  $U = 0$  and  $J < 0$  where the lowest energy states (2) are  $N + 1$  degenerate ( $p = N$  and  $q = 0$ ) and  $\vec{T}_i$  is a  $N/2$  pseudospin operator. At second order of perturbation theory in  $|J| \gg t$ , we find a pseudospin- $N/2$  antiferromagnetic  $SU(2)$  Heisenberg chain:  $\mathcal{H}_{\text{eff}} = J_o \sum_i \vec{T}_i \cdot \vec{T}_{i+1}$  with  $J_o = 8t^2/N(2N+1)|J|$ . As it is well known, the physics of the latter model strongly depends on the parity of  $N$ .<sup>3</sup> When  $N$  is odd, the phase is gapless with one gapless bosonic mode while the phase is fully gapped (Haldane gap) in the even  $N$  case. We thus find the emergence of an  $SU(2)$  topological (Haldane) phases when  $N$  is even, where the  $SU(2)$  symmetry stems from the orbital sector defined by the two  $g, e$  states of alkaline-earth atoms. We dub this phase, *Haldane-orbital phase* (HO), since this phase is different from the Haldane phase for (nuclear) spin degrees of freedom (see Fig. 1 where a cartoon represents this phase for  $N = 2$ ). These Haldane phases are known to be topologically protected by a symmetry only when  $N/2$  is odd, i.e.,  $I = 1/2, 5/2, 9/2$  for the simplest values.<sup>6</sup>

A last interesting line is  $J = 2NU/(N+2) > 0$  with even  $N$ . The lowest energy states correspond then to  $p = q = N/2$  ( $N$  fermions which are orbital singlets) and transform into the  $SU(N)$  self-conjugate representations described by a Young

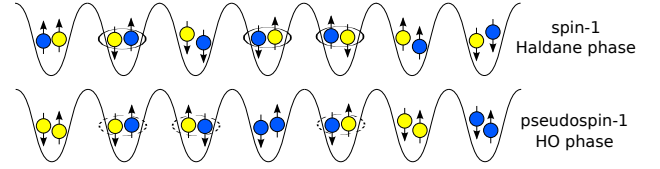


FIG. 1: (Color online) For  $N = 2$ : the blue (respectively yellow) atoms are in the  $g$  (respectively  $e$ ) state. The Haldane phases for nuclear spin and orbital (HO) degrees of freedom correspond to a (pseudo)spin-1 Heisenberg chain. These phases present a dilute order<sup>5</sup> that can be seen here as an alternation of sites with total (pseudo)spin component  $\pm 1$ , diluted with an arbitrary number of sites with (pseudo)spin component 0.

tableau with two columns and  $N/2$  rows. At second order of perturbation theory in  $|U| \gg t$ , we find an  $SU(N)$  Heisenberg spin chain in such representation. For  $N = 2$ , the spin-1 Haldane phase is formed among the nuclear spins and, in this respect, is intrinsically different from the HO phase with  $N = 2$  (see Fig. 1). When  $N > 2$ , the physical properties of the ground state of the latter  $SU(N)$  magnet are not known. A confinement of spinons is expected from the general classification of Ref.19, and a non-degenerate gapful phase has been predicted in the large  $N$  limit.<sup>20</sup>

**Valence-bond-solid (VBS) approach.** To get a good insight into the properties of the ground state of this  $SU(N)$  Heisenberg chain, we construct a series of model ground states, the VBS states,<sup>21</sup> whose parent Hamiltonian is close to the original one in question. We start from a pair of the self-conjugate representations (characterized by a Young tableau with *one* column and  $N/2$  rows) on each site and create maximally-entangled pairs between adjacent sites (see Fig. 2). Last, we obtain the model VBS state by projecting the tensor-product states on each site onto the desired self-conjugate representation.<sup>22</sup> We explicitly constructed the matrix-product representation

$$\sum_{\{m_i\}} A_1(m_1)A_2(m_2) \cdots A_i(m_i) \cdots |m_1, m_2, \dots, m_i, \dots\rangle \quad (3)$$

of such a state for  $N = 4$  (with the dimension of the  $A$ -matrices being 6 and  $\{m_i\}$  labeling the 20 physical states at each site) to obtain ‘spin-spin’ correlations exponentially decaying with correlation length  $1/\ln 5 \approx 0.6213$ . The parent Hamiltonian which supports the above VBS state as the exact ground state is given by

$$\mathcal{H}_{\text{VBS}} = J_s \sum_i \left\{ S_i^A S_{i+1}^A + \frac{13}{108} (S_i^A S_{i+1}^A)^2 + \frac{1}{216} (S_i^A S_{i+1}^A)^3 \right\}, \quad (4)$$

where  $S_i^A$  denote the  $SU(4)$  spin operators in the 20-dimensional representation. We observe that model (4) is not very far from the original pure Heisenberg Hamiltonian (with spin exchange  $J_s$ ) obtained by the  $t/U$ -expansion. This strongly suggests that an  $SU(4)$  topological phase is stabilized in the strong-coupling regime with the emergent edge states belonging to the 6-dimensional representation of  $SU(4)$ . From

the structure of the edge states, one can easily see that the VBS state found above belongs to one of the  $N$  topological classes protected by  $SU(N)$  symmetry.<sup>9</sup>

It is interesting to see if it is robust even in the absence of  $SU(N)$  symmetry or not. Recently, it has been demonstrated that the even-fold degenerate structure in the entanglement spectrum is a fingerprint of the topological Haldane phase protected by inversion symmetry.<sup>2</sup> One may think that the  $N = 4$  VBS state obtained above represents a topologically robust state since its six finite entanglement eigenvalues are all degenerate. However, this degeneracy is accidental and the corresponding topological phase is protected only by high symmetries. In fact, if the phase is protected by the link-inversion symmetry, the unitary matrix  $U_I$  satisfying<sup>23</sup>

$$A^T(m) = e^{i\theta_I} U_I^\dagger A(m) U_I \quad (5)$$

should be anti-symmetric.<sup>2</sup> For the  $N = 4$  VBS state,  $U_I^T = U_I$  and the state becomes trivial only in the presence of such an elementary symmetry as the link-inversion. Since the dimension of the self-conjugate representation is rapidly growing (1764 for  $SU(8)$ ), it is not practical to explicitly construct the MPS for  $N \geq 6$ . However, it can be shown that the symmetry of  $U_I$  is determined solely by that of the maximally-entangled (singlet) pair; if the maximally-entangled pair is anti-symmetric with respect to the interchange of the two constituent states,  $U_I$  is anti-symmetric and the corresponding MPS remains topological even without high symmetries. By investigating the form of the maximally-entangled pair, we conclude that for  $N = 2(2n + 1)$  ( $n \geq 0$ ) the VBS state is in the stable topological phase protected by inversion symmetry and not for  $N = 4n$ . In particular, as in the HO case, we expect that the gapped  $SU(N)$  phase realized along the line  $J = 2NU/(N + 2)$  is SPT when  $I = 1/2, 5/2, 9/2, \text{etc.}$

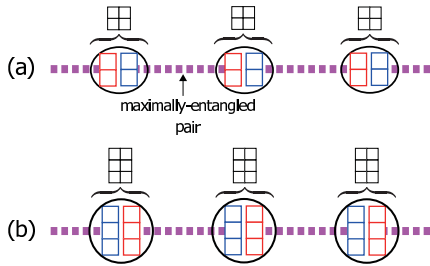


FIG. 2: (Color online)  $SU(N)$  VBS states are constructed out of a pair of self-conjugate representations at each site. Dashed lines denote maximally-entangled pairs. (a)  $SU(4)$  with 20-dimensional representation and (b)  $SU(6)$  with 175-dimensional representation.

**Low-energy approach.** The low-energy effective field theory of the lattice model (1) is derived by expressing the standard continuum limit of the lattice fermionic operators  $c_{l\alpha i}$  in terms of  $2N$  left- and right-moving  $L_{l\alpha}, R_{l\alpha}$  Dirac fermions:<sup>24</sup>  $c_{l\alpha i} \sim R_{l\alpha} e^{ik_F x} + L_{l\alpha} e^{-ik_F x}$ ,  $k_F = \pi/(2a_0)$  being the Fermi momentum and  $x = ia_0$  ( $a_0$  being the lattice spacing). The continuum Hamiltonian then takes the form of  $2N$  Dirac fermions coupled with marginal four-fermions interactions. The low-energy properties of the latter model

are determined by means of a one-loop renormalization group (RG) analysis. This analysis has been done in the  $N = 2$  case in Refs.25,26. For instance, four different fully gapped Mott-insulating phases have been found when  $J = J_z$ . On top of the two-fold degenerate phases with SP- and CDW orderings, the pseudospin-1 HO and the spin-1 Haldane phases, which have been identified in the strong-coupling analysis, persist in the weak-coupling regime as well.

The RG analysis in the general  $N > 2$  case is much more involved. In some regions of the phase diagram, we find that the one-loop RG flow is attracted along two isotropic rays with  $SO(4N)$  symmetry which is the maximal continuous symmetry achievable for  $2N$  Dirac fermions. These highly-symmetric rays signal the emergence of the CDW and SP phases in the general  $N$  case.<sup>17</sup> In sharp contrast to  $N = 2$ , the one-loop RG flow for  $N > 2$  features a region with no symmetry restoration in the infrared (IR) limit. There is a separation of the energy scales and one of the perturbations, which depends only on the  $SU(N)$  spin degrees of freedom, reaches the strong-coupling regime faster than the others. A spin gap  $\Delta_s$  is thus formed among the nuclear spin degrees of freedom. Below the energy scale of the spin gap  $E \ll \Delta_s$ , the dominant part of the effective interacting Hamiltonian only involves the remaining charge and orbital degrees of freedom and, when  $J = J_z$ , reads as follows:

$$\mathcal{H}_{int}^{\text{eff}} \simeq \lambda (\text{Tr} g)^2 + \mu \cos \left( \sqrt{8\pi K_c / N} \Phi_c \right), \quad (6)$$

where  $\Phi_c$  is a Bose field which accounts for the  $U(1)_c$  charge degrees of freedom, and  $K_c$  is its Luttinger parameter.<sup>24</sup> The low-energy properties of the charge degrees of freedom are captured by the sine-Gordon model at  $\beta^2 = 8\pi K_c / N$  and we expect that the charge sector is gapped away since  $K_c < N$ . The IR properties of model (6) thus depend only the orbital degrees of freedom and are described by the  $SU(2)_N$  conformal field theory (CFT) perturbed by its spin-1 operator  $(\text{Tr} g)^2$  ( $g$  being the  $SU(2)$  matrix) with scaling dimension  $4/(N + 2)$ . In this respect, the resulting low-energy field theory is exactly that of the spin- $N/2$   $SU(2)$  Heisenberg chain derived by Affleck and Haldane using the non-Abelian bosonization approach.<sup>27</sup> We thus deduce the emergence of a Haldane-gap phase when  $N$  is even, i.e. for general half-integer nuclear spins. The resulting Haldane phase is identified with the HO phase which have been found already by the strong-coupling approach and is a collective singlet state formed among the orbital degrees of freedom. In stark contrast, within the weak-coupling approach, we could see no evidence for a similar  $SU(N)$  topological phase of the nuclear spins for  $N > 2$ . In particular, along the  $J = 2NU/(N + 2) > 0$  line ( $N$  being even), a SP phase is found in the RG approach instead of an  $SU(N)$  topological phase expected from the strong-coupling argument. Therefore, a quantum phase transition necessarily occurs at intermediate couplings and it is tempting to conjecture that the quantum critical point is described by an  $SU(N)_2$  CFT. The situation may be understood as follows. First, by symmetry, one can determine the low-energy field theory which is valid in the vicinity of this  $SU(N)_2$  quantum critical point:  $\mathcal{H}_{\text{eff}} \simeq \mathcal{H}_{SU(N)_2} + (g - g_c) |\text{Tr} G|^2$ ,  $G$  being

the  $SU(N)_2$  primary field which belongs to the fundamental representation of  $SU(N)$ . Following Ref.28, the nature of the phases of the latter model can be inferred from a semiclassical approach; when  $g < g_c$ , one has  $\langle \text{Tr} G \rangle \neq 0$ , and the ground state is two-fold degenerate as a consequence of broken translation symmetry ( $G \rightarrow -G$ ). This may be identified with the SP phase found in the weak-coupling limit. On the strong-coupling side  $g > g_c$ , the semiclassical analysis now gives an  $SU(N)$  matrix with the constraint  $\text{Tr} G = 0$ , and the phase is translationally invariant. The resulting effective field theory is known to be the Grassmannian sigma model on  $U(N)/[U(N/2) \times U(N/2)]$  manifold with a  $\theta = 2\pi$ , topological theta term, which is massive.<sup>28</sup> The latter is known to be the semiclassical field theory of the  $SU(N)$  Heisenberg spin chain in self-conjugate representations with two columns.<sup>20</sup> Therefore, we conclude that the  $SU(N)$  topological phase, identified within the VBS approach based on the strong-coupling Hamiltonian, emerges for  $g > g_c$ .

**DMRG calculations.** A density-matrix renormalization group<sup>29</sup> (DMRG) is clearly called for to shed light on the existence of this quantum phase transition for moderate couplings. We have rewritten model (1) as a  $N$ -leg Hubbard ladder with additional rung interactions in order to get more efficient simulations. Typically, we had to keep up to 1600 or 3200 states for convergence with  $N = 2$  and  $N = 4$  respectively, and to get a discarded weight below  $10^{-5}$ , and we use open boundary conditions.

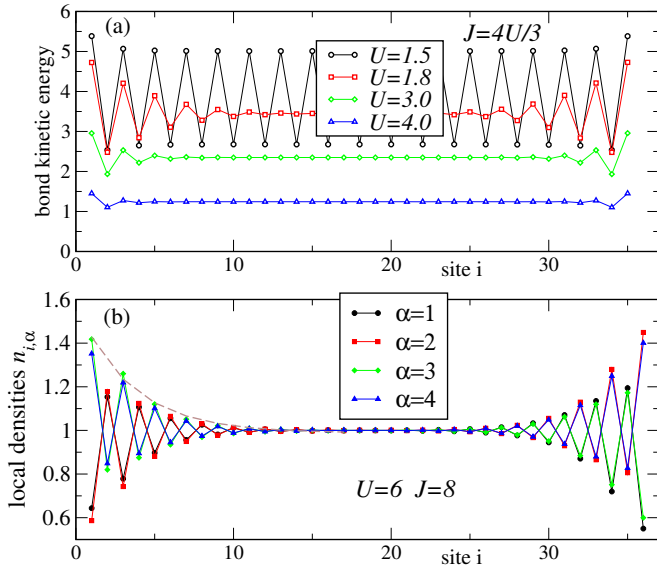


FIG. 3: (Color online) (a) Bond kinetic energy for various  $U > 0$  along the line  $J = 4U/3$  ( $N = 4$ ) on  $L = 36$  chain, the hopping  $t$  is fixed to one. (b) local densities  $n_{i,\alpha}$  (see text for definition) in the topological phase ( $U = 6$  and  $J = 8$ ) showing the existence of edge states. The dashed line is an exponential fit with a correlation length of around 3 lattice spacings.

In Fig. 3(a), we plot the bond kinetic energy for various  $U > 0$  using  $N = 4$  along the special line  $J = 2NU/(N + 2) = 4U/3$  for a chain of length  $L = 36$ . As  $U$  increases, we clearly see a quantum phase transition from a SP phase to

a uniform non-degenerate one. This is in agreement with our conjecture based on the low-energy analysis and the strong coupling one. In order to confirm that the non-degenerate phase corresponds to an  $SU(4)$  topological phase, we plot in Fig. 3(b) the local densities  $n_{i,\alpha} = \sum_l c_{l\alpha,i}^\dagger c_{l\alpha,i}$  for each flavor  $\alpha = 1, \dots, 4$  in this phase. Clearly, edge states are present with an exponential profile and a correlation length of the order of 3 lattice spacings for  $U = 6$  and  $J = 8$  (see fit). Moreover, DMRG data randomly shows different kinds of edge states, in the sense that for a given edge, two flavors out of four give the same local densities. The same phenomenon occurs in the Haldane phase of spin-1 chain where a simulation done in the  $S_z^{\text{tot}} = 0$  sector will generate arbitrarily an edge state with  $S_z = \pm 0.5$  at one boundary. Now, since there are 6 ways to choose two flavors out of four, our numerical data confirm that edge states are 6-fold degenerate in agreement with the VBS prediction that they belong to 6-dimensional  $SU(4)$  representation. We also observe that these edge states are absent in the SP phase. The precise location of the phase transition and its universality class require extensive large-scale simulations and are beyond the scope of the Letter.

Now, we turn to the illustration of the HO phase for  $N = 2$  since the case  $N = 4$  would be quite demanding numerically.<sup>30</sup> In Fig. 4(a), we show the presence of edge states in a finite chain of length  $L = 64$ , similarly to what is found in the Haldane phase of the spin-1 chain. In Fig. 4(b), we also plot pseudo-spin correlations (taken from the middle of the chain) that are short-ranged and almost identical to spin correlations measured in a  $S = 1$  Heisenberg chain. Overall, we confirm the existence of the HO phase in this region.

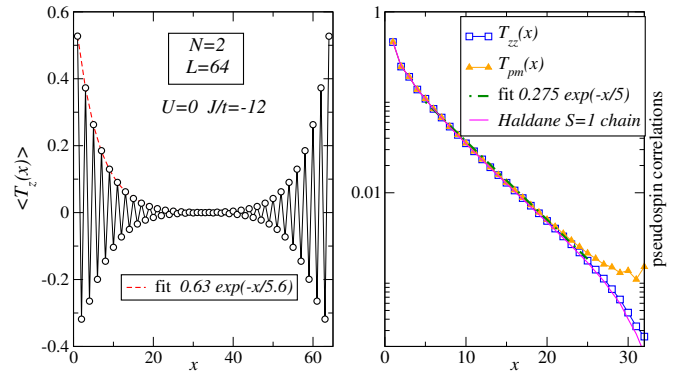


FIG. 4: (Color online)  $N = 2$  DMRG numerical data for a chain of length  $L = 64$  with  $U = 0$  and  $J/t = -12$  (a) local average  $\langle T_z(x) \rangle$  vs position  $x$  in the ground-state with total  $T_z = 1$  showing evidence of localized edge states; (b) pseudospin correlations  $T_{zz}(x) = \langle T_z(L/2)T_z(L/2 + x) \rangle$  and  $T_{pm}(x) = \langle T_+(L/2)T_-(L/2 + x) \rangle / 2$  in the ground-state with  $T_z = 0$  exhibit an  $SU(2)_o$  symmetry, are short-ranged and almost identical to spin correlations  $\langle S_z(L/2)S_z(L/2 + x) \rangle$  measured in  $S = 1$  Heisenberg chain. Bulk and edge correlation lengths are close to 5 lattice spacings.

The authors would like to thank E. Boulat and T. Koffel for useful discussions. Numerical simulations were performed at CALMIP.

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